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$A_r(\Omega)$-Weighted Imbedding Inequalities for A-Harmonic Tensors

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Abstract. We prove the basic $A_r(\Omega)$-weighted imbedding inequalities for A-harmonic tensors. These results can be used to estimate the integrals for A-harmonic tensors and to study the integrability of A-harmonic tensors and the properties of the homotopy operator $T\colon C^\infty(D, \wedge^l) \to C^\infty(D, \wedge^{l-1})$.

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1. Introduction

Let $e_1, e_2, \ldots, e_n$ denote the standard unit basis of $\mathbf{R}^n$, $n \geqslant 2$, and $\mathbf{R} = \mathbf{R}^1$. Assume that $\wedge^l = \wedge^l(\mathbf{R}^n)$ is the linear space of l-vectors, spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots e_{i_l}$, corresponding to all ordered l-tuples $I = (i_1, i_2, \ldots, i_l)$, $1 \leqslant i_1 < i_2 < \cdots < i_l \leqslant n$, $l = 0, 1, \ldots, n$. The Grassman algebra $\wedge = \bigoplus \wedge^l$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha^I e_I \in \wedge$ and $\beta = \sum \beta^I e_I \in \wedge$, the inner product in $\wedge$ is given by $\langle \alpha, \beta \rangle = \sum \alpha^I \beta^I$ with summation over all l-tuples $I = (i_1, i_2, \ldots, i_l)$ and all integers $l = 0, 1, \ldots, n$. We define the Hodge star operator $\star : \wedge \to \wedge$ by the rule $\star 1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n$ and $\alpha \wedge \star\beta = \beta \wedge \star\alpha = \langle \alpha, \beta \rangle (\star 1)$ for all $\alpha, \beta \in \wedge$. The norm of $\alpha \in \wedge$ is given by the formula $|a|^2 = \langle \alpha, \alpha \rangle = \star(\alpha \wedge \star\alpha) \in \wedge^0 = \mathbf{R}$. The Hodge star is an isometric isomorphism on $\wedge$ with $\star : \wedge^l \to \wedge^{n-l}$ and $\star\star\, (-1)^{l(n-l)}\colon \wedge^l \to \wedge^l$.

We always assume that Ω is a bounded domain in $\mathbf{R}^n$ throughout this paper. Balls are denoted by B and σB is the ball with the same center as B and with $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$. We do not distinguish the balls from cubes throughout this paper. The n-dimensional Lebesgue measure of a set $E \subseteq \mathbf{R}^n$ is denoted by $|E|$. We call w a weight if $w \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and $w > 0$ a.e. For $0 < p < \infty$, we denote the weighted L^p-norm of a measurable function f over E by

$$\|f\|_{p,E,w} = \left(\int_E |f(x)|^p w(x)\, \mathrm{d}x \right)^{1/p}.$$

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A differential l-form w on Ω is a de Rham current (see [9, Chapter III]) on Ω with values in $\wedge^l(\mathbf{R})$. We use $D'(\Omega, \wedge^l)$ to denote the space of all differential l-forms and $L^p(\Omega, \wedge^l)$ to denote the l-forms $w(x) = \sum_I w_I(x)\,\mathrm{d}x_I = \sum w_{i_1 i_2 \ldots i_l}(x)\,\mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \cdots \wedge \mathrm{d}x_{i_l}$ with $w_I \in L^p(\Omega, \mathbf{R})$ for all ordered l-tuples I. Thus $L^p(\Omega, \wedge^l)$ is a Banach space with norm

$$\|w\|_{p,\Omega} = \left(\int_\Omega |w(x)|^p\,\mathrm{d}x\right)^{1/p} = \left(\int_\Omega \left(\sum_I |w_I(x)|^2\right)^{p/2} \mathrm{d}x\right)^{1/p}.$$

Similarly, $W_p^1(\Omega, \wedge^l)$ are those differential l-forms on Ω whose coefficients are in $W_p^1(\Omega, \mathbf{R})$. The notations $W_{p,\mathrm{loc}}^1(\Omega, \mathbf{R})$ and $W_{p,\mathrm{loc}}^1(\Omega, \wedge^l)$ are self-explanatory. We denote the exterior derivative by $\mathrm{d}: D'(\Omega, \wedge^l) \to D'(\Omega, \wedge^{l+1})$ for $l = 0, 1, \ldots, n$. Its formal adjoint operator $\mathrm{d}^\star : D'(\Omega, \wedge^{l+1}) \to D'(\Omega, \wedge^l)$ is given by $\mathrm{d}^\star = (-1)^{nl+1} \star \mathrm{d}\star$ on $D'(\Omega, \wedge^{l+1})$, $l = 0, 1, \ldots, n$.

T. Iwaniec and A. Lutoborski prove the following result in [7]: Let $D \subset \mathbf{R}^n$ be a bounded, convex domain. To each $y \in D$ there corresponds a linear operator $K_y : C^\infty(D, \wedge^l) \to C^\infty(D, \wedge^{l-1})$ defined by

$$(K_y w)(x; \xi_1, \ldots, \xi_l) = \int_0^1 t^{l-1} w(tx + y - ty; x - y, \xi_1, \ldots, \xi_{l-1})\,\mathrm{d}t$$

are the decomposition

$$w = \mathrm{d}(K_y w) + K_y(\mathrm{d}w).$$

Then, T. Iwaniec and A. Lutoborski introduce a homotopy operator $T\colon C^\infty(D, \wedge^l) \to C^\infty(D, \wedge^{l-1})$ by averaging K_y over all points y in D

$$Tw = \int_D \varphi(y) K_y w\,\mathrm{d}y, \tag{1.1}$$

where $\varphi \in C_0^\infty(D)$ is normalized by $\int_D \varphi(y)\,\mathrm{d}y = 1$, and prove the following imbedding inequalities for differential forms.

THEOREM A. *Let $u \in L_{\mathrm{loc}}^s(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a differential form in a bounded domain $\Omega \subset \mathbf{R}^n$. Assume that F is any convex subset such that $\operatorname{supp}\varphi \subset F \subset \Omega$, where φ from $C_0^\infty(\Omega)$ is normalized by $\int_\Omega \varphi(y)\,\mathrm{d}y = 1$. Then*

(i) $\|Tu\|_{s,F} \leqslant C\,\mathrm{diam}(F)\|u\|_{s,F}$;
(ii) $\|\mathrm{d}(Tu)\|_{s,F} \leqslant \|u\|_{s,F} + C\,\mathrm{diam}(F)\|\mathrm{d}u\|_{s,F}$,

where $C = 2^n \sigma_{n-1} \nu(\Omega)$, σ_{n-1} denotes the surface area of the unit sphere in $\mathbf{R}^n$ and

$$\nu(\Omega) = \frac{(\mathrm{diam}(\Omega))^{n+1}}{\int_\Omega \mathrm{dist}(y, \partial\Omega)\,\mathrm{d}y}.$$

The imbedding inequalities have been playing important roles in developing the L^p theory of differential forms, see [7]. In this paper, we prove the $A_r(\Omega)$-weighted imbedding inequalities for A-harmonic tensors.

Many interesting results (see [1–4, 7, 8]) have been established in the study of the p-harmonic equation

$$d^\star(|du|^{p-2}du) = 0$$

and the A-harmonic equation

$$d^\star A(x, dw) = 0 \tag{1.2}$$

for differential forms, where $A : \Omega \times \wedge^l(\mathbf{R}^n) \to \wedge^l(\mathbf{R}^n)$ satisfies the following conditions:

$$|A(x, \xi)| \leqslant a|\xi|^{p-1} \quad \text{and} \quad \langle A(x, \xi), \xi\rangle \geqslant |\xi|^p \tag{1.3}$$

for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbf{R}^n)$. Here $a > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with (1.2). A solution to (1.2) is an element of the Sobolev space $W^1_{p,\mathrm{loc}}(\Omega, \wedge^{l-1})$ such that

$$\int_\Omega \langle A(x, dw), d\varphi\rangle = 0$$

for all $\varphi \in W^1_p(\Omega, \wedge^{l-1})$ with compact support.

DEFINITION 1.4. We call u an A-harmonic tensor in Ω if u satisfies the A-harmonic equation (1.2) in Ω.

A differential l-form $u \in D'(\Omega, \wedge^l)$ is called a closed form if $du = 0$ in Ω. Similarly, a differential $(l + 1)$-form $v \in D'(\Omega, \wedge^{l+1})$ is called a coclosed form if $d^\star v = 0$. Clearly, the A-harmonic equation is not affected by adding a closed form to w. Therefore, any type of estimates about u must be modulo a closed form.

2. Local Weighted Imbedding Inequalities

DEFINITION 2.1. A weight $w(x)$ is called an A_r-weight for some $r > 1$ in a domain Ω, write $w \in A_r(\Omega)$, if $w(x) > 0$ a.e., and

$$\operatorname{supp}_B \left(\frac{1}{|B|}\int_B w\, dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{(r-1)} < \infty \tag{2.2}$$

for any ball $B \subset \Omega$.

See [5] and [6] for properties of $A_r(\Omega)$-weights. We will need the following generalized Hölder inequality.

LEMMA 2.3. *Let* $0 < \alpha < \infty$, $0 < \beta < \infty$ *and* $s^{-1} = \alpha^{-1} + \beta^{-1}$. *If* f *and* g *are measurable functions on* $\mathbf{R}^n$, *then*

$$\|fg\|_{s,\Omega} \leqslant \|f\|_{\alpha,\Omega} \cdot \|g\|_{\beta,\Omega}$$

for any $\Omega \subset \mathbf{R}^n$.

We also need the following lemma [5].

LEMMA 2.4. *If* $w \in A_r(\Omega)$, *then there exist constants* $\beta > 1$ *and* C, *independent of* w, *such that*

$$\|w\|_{\beta,B} \leqslant C|B|^{(1-\beta)/\beta}\|w\|_{1,B}$$

for all balls $B \subset \mathbf{R}^n$.

The following weak reverse Hölder inequality appears in [8].

LEMMA 2.5. *Let* u *be an* A-*harmonic tensor in* Ω, $\rho > 1$ *and* $0 < s, t < \infty$. *Then there exists a constant* C, *independent of* u, *such that*

$$\|u\|_{s,B} \leqslant C|B|^{(t-s)/st}\|u\|_{t,\rho B}$$

for all balls or cubes B *with* $\rho B \subset \Omega$.

Now we prove the following weighted imbedding inequality for A-harmonic tensors and the homotopy operator T.

THEOREM 2.6. *Let* $u \in L^s_{\text{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, *be an* A-*harmonic tensor in a bounded domain* $\Omega \subset \mathbf{R}^n$ *and* $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ *be a homotopy operator defined in* (1.1). *Assume that* $\rho > 1$ *and* $w \in A_r(\Omega)$ *for some* $r > 1$. *Then, for any ball* B *such that* $\operatorname{supp}\varphi \subset B \subset \rho B \subset \Omega$, *where* φ *from* $C_0^\infty(B)$ *is normalized by* $\int_B \varphi(y)\,\mathrm{d}y = 1$, *there exists a constant* C, *independent of* u, *such that*

$$\left(\int_B |Tu|^s w^\alpha \,\mathrm{d}x\right)^{1/s} \leqslant C\,\operatorname{diam}(B)\left(\int_{\rho B} |u|^s w^\alpha \,\mathrm{d}x\right)^{1/s} \tag{2.7}$$

for any real number α *with* $0 < \alpha \leqslant 1$.

Proof. We first show that (2.7) holds for $0 < \alpha < 1$. Let $t = s/(1-\alpha)$. Using Lemma 2.3, we have

$$\begin{aligned}\left(\int_B |Tu|^s w^\alpha \,\mathrm{d}x\right)^{1/s} &= \left(\int_B \left(|Tu| w^{\alpha/s}\right)^s \mathrm{d}x\right)^{1/s} \\ &\leqslant \|Tu\|_{t,B}\left(\int_B w^{t\alpha/(t-s)}\,\mathrm{d}x\right)^{(t-s)/st} \\ &= \|Tu\|_{t,B}\left(\int_B w \,\mathrm{d}x\right)^{\alpha/s}. \end{aligned} \tag{2.8}$$

By Theorem A, we have

$$\|Tu\|_{t,B} \leqslant C_1 \operatorname{diam}(B)\|u\|_{t,B}. \tag{2.9}$$

Choose $m = s/(1+\alpha(r-1))$, then $m < s$. Substituting (2.9) into (2.8) and using Lemma 2.5, we have

$$\begin{aligned}\left(\int_B |Tu|^s w^\alpha \,\mathrm{d}x\right)^{1/s} &\leqslant C_1 \operatorname{diam}(B)\|u\|_{t,B}\left(\int_B w\,\mathrm{d}x\right)^{\alpha/s}\\ &\leqslant C_2 \operatorname{diam}(B)|B|^{(m-t)/mt}\|u\|_{m,\rho B}\left(\int_B w\,\mathrm{d}x\right)^{\alpha/s}. \end{aligned}\tag{2.10}$$

Using Lemma 2.3 with $1/m = 1/s + (s-m)/sm$, we obtain

$$\begin{aligned}\|u\|_{m,\rho B} &= \left(\int_{\rho B} |u|^m \,\mathrm{d}x\right)^{1/m}\\ &= \left(\int_{\rho B} \left(|u| w^{\alpha/s} w^{-\alpha/s}\right)^m \mathrm{d}x\right)^{1/m}\\ &\leqslant \left(\int_{\rho B} |u|^s w^\alpha \,\mathrm{d}x\right)^{1/s}\left(\int_{\rho B}\left(\frac{1}{w}\right)^{1/(r-1)} \mathrm{d}x\right)^{\alpha(r-1)/s}\end{aligned}\tag{2.11}$$

for all balls B with $\rho B \subset \Omega$. Substituting (2.11) into (2.10), we obtain

$$\begin{aligned}&\left(\int_B |Tu|^s w^\alpha \,\mathrm{d}x\right)^{1/s}\\ &\quad\leqslant C_2 \operatorname{diam}(B)|B|^{(m-t)/mt}\left(\int_{\rho B} |u|^s w^\alpha \,\mathrm{d}x\right)^{1/s}\times\\ &\quad\quad\times\left(\int_B w\,\mathrm{d}x\right)^{\alpha/s}\left(\int_{\rho B}\left(\frac{1}{w}\right)^{1/(r-1)} \mathrm{d}x\right)^{\alpha(r-1)/s}.\end{aligned}\tag{2.12}$$

Now $w \in A_r(\Omega)$ yields

$$\begin{aligned}\|w\|_{1,B}^{\alpha/s}\cdot\|1/w\|_{1/(r-1),\rho B}^{\alpha/s} &\leqslant \left(\left(\int_{\rho B} w\,\mathrm{d}x\right)\left(\int_{\rho B}(1/w)^{1/(r-1)}\,\mathrm{d}x\right)^{r-1}\right)^{\alpha/s}\\ &= \left(|\rho B|^r\left(\frac{1}{|\rho B|}\int_{\rho B} w\,\mathrm{d}x\right)\times\right.\\ &\quad\left.\times\left(\frac{1}{|\rho B|}\int_{\rho B}\left(\frac{1}{w}\right)^{1/(r-1)}\mathrm{d}x\right)^{r-1}\right)^{\alpha/s}\\ &\leqslant C_3|B|^{\alpha r/s}.\end{aligned}\tag{2.13}$$

Combining (2.13) and (2.12), we find that

$$\left(\int_B |Tu|^s w^\alpha \,\mathrm{d}x\right)^{1/s} \leqslant C_4 \operatorname{diam}(B)\left(\int_{\rho B} |u|^s w^\alpha \,\mathrm{d}x\right)^{1/s}\tag{2.14}$$

for all balls B with $\rho B \subset \Omega$. We have proved that (2.7) is true if $0 < \alpha < 1$.

Next, we prove (2.7) is true for $\alpha = 1$, that is, we need to show that

$$\|Tu\|_{s,B,w} \leqslant C\,\mathrm{diam}(B)\|u\|_{s,\rho B,w}. \tag{2.15}$$

By Lemma 2.4, there exist constants $\beta > 1$ and $C_5 > 0$, such that

$$\|w\|_{\beta,B} \leqslant C_5|B|^{(1-\beta)/\beta}\|w\|_{1,B} \tag{2.16}$$

for any cube or any ball $B \subset \mathbf{R}^n$. Choose $t = s\beta/(\beta-1)$, then $1 < s < t$ and $\beta = t/(t-s)$. Since $1/s = 1/t + (t-s)/st$, by Lemma 2.3, Theorem A and (2.16), we have

$$\begin{aligned}
\left(\int_B |Tu|^s w\,\mathrm{d}x\right)^{1/s} &= \left(\int_B (|Tu|w^{1/s})^s\,\mathrm{d}x\right)^{1/s}\\
&\leqslant \left(\int_B |Tu|^t\,\mathrm{d}x\right)^{1/t}\left(\int_B (w^{1/s})^{st/(t-s)}\,\mathrm{d}x\right)^{(t-s)/st}\\
&\leqslant C_6\|Tu\|_{t,B}\cdot\|w\|_{\beta,B}^{1/s}\\
&\leqslant C_6\,\mathrm{diam}(B)\|u\|_{t,B}\cdot\|w\|_{\beta,B}^{1/s}\\
&\leqslant C_7\,\mathrm{diam}(B)|B|^{(1-\beta)/\beta s}\|w\|_{1,B}^{1/s}\cdot\|u\|_{t,B}\\
&\leqslant C_7\,\mathrm{diam}(B)|B|^{-1/t}\|w\|_{1,B}^{1/s}\cdot\|u\|_{t,B}.
\end{aligned} \tag{2.17}$$

Let $m = s/r$. From Lemma 2.4, we find that

$$\|u\|_{t,B} \leqslant C_8|B|^{(m-t)/mt}\|u\|_{m,\rho B}. \tag{2.18}$$

Lemma 2.3 yields

$$\begin{aligned}
\|u\|_{m,\rho B} &= \left(\int_{\rho B} (|u|w^{1/s}w^{-1/s})^m\,\mathrm{d}x\right)^{1/m}\\
&\leqslant \left(\int_{\rho B} |u|^s w\,\mathrm{d}x\right)^{1/s}\left(\int_{\rho B}\left(\frac{1}{w}\right)^{1/(r-1)}\mathrm{d}x\right)^{(r-1)/s}
\end{aligned} \tag{2.19}$$

for all balls B with $\rho B \subset \Omega$. Note that $w \in A_r(\Omega)$. Then

$$\begin{aligned}
\|w\|_{1,B}^{1/s}\cdot\|1/w\|_{1/(r-1),\rho B}^{1/s} &\leqslant \left(\left(\int_{\rho B} w\,\mathrm{d}x\right)\left(\int_{\rho B}(1/w)^{1/(r-1)}\,\mathrm{d}x\right)^{r-1}\right)^{1/s}\\
&= \left(|\rho B|^r\left(\frac{1}{|\rho B|}\int_{\rho B} w\,\mathrm{d}x\right)\times\right.\\
&\quad\left.\times\left(\frac{1}{|\rho B|}\int_{\rho B}\left(\frac{1}{w}\right)^{1/(r-1)}\mathrm{d}x\right)^{r-1}\right)^{1/s}\\
&\leqslant C_9|B|^{r/s}.
\end{aligned} \tag{2.20}$$

Combining (2.17), (2.18), (2.19) and (2.20), we have

$$\begin{aligned}\|Tu\|_{s,B,w} &\leqslant C_{10}\,\mathrm{diam}(B)|B|^{-1/t}\|w\|_{1,B}^{1/s}|B|^{(m-t)/mt}\|u\|_{m,\rho B}\\ &\leqslant C_{10}\,\mathrm{diam}(B)|B|^{-1/m}\|w\|_{1,B}^{1/s}\cdot\|1/w\|_{1/(r-1),\rho B}^{1/s}\|u\|_{s,\rho B,w}\\ &\leqslant C_{11}\,\mathrm{diam}(B)\|u\|_{s,\rho B,w}\end{aligned}$$

for all balls B with $\rho B\subset\Omega$. Hence, (2.15) holds. The proof of Theorem 2.6 is completed. □

THEOREM 2.21. *Let* $\in L^s_{\mathrm{loc}}(\Omega,\wedge^l)$, $l=1,2,\dots,n$, $1<s<\infty$, *be an* A*-harmonic tensor in a bounded domain* $\Omega\subset\mathbf{R}^n$ *such that* $du\in L^s_{\mathrm{loc}}(\Omega,\wedge^{l+1})$ *and* $T:C^\infty(\Omega,\wedge^l)\to C^\infty(\Omega,\wedge^{l-1})$ *be a homotopy operator defined in* (1.1). *Assume that* $\sigma>1$ *and* $w\in A_r(\Omega)$ *for some* $r>1$. *Then, for any ball* B *such that* $\mathrm{supp}\,\varphi\subset B\subset\Omega$, *where* φ *from* $C_0^\infty(B)$ *is normalized by* $\int_B\varphi(y)\,\mathrm{d}y=1$, *we have*

$$\left(\int_B|\mathrm{d}(Tu)|^s w^\alpha\mathrm{d}x\right)^{1/s}\leqslant C|B|^{(1-\alpha)/s}\left(\int_{\sigma B}|u|^s w^\alpha\,\mathrm{d}x\right)^{1/s}\tag{2.22}$$

for all balls B *with* $\sigma B\subset\Omega$ *and any real number* α *with* $0<\alpha\leqslant1$.

Proof. First, we assume that $0<\alpha<1$. Let $t=s/(1-\alpha)$. From Caccioppoli-type estimate for A-harmonic tensors, we know that there exists a constant C_1, independent of u, such that

$$\|\mathrm{d}u\|_{t,B}\leqslant C_1\,\mathrm{diam}(B)^{-1}\|u\|_{t,\sigma B}\tag{2.23}$$

for any A-harmonic tensor u in Ω and all balls or cubes B with $\sigma B\subset\Omega$, where $\sigma>1$. Now let $m=s/(1+\alpha(r-1))$. Using Lemma 2.3, (ii) in Theorem A, (2.23) and Lemma 2.5, we have

$$\begin{aligned}\left(\int_B|\mathrm{d}(Tu)|^s w^\alpha\,\mathrm{d}x\right)^{1/s} &= \left(\int_B\left(|\mathrm{d}(Tu)|w^{\alpha/s}\right)^s\mathrm{d}x\right)^{1/s}\\ &\leqslant \|\mathrm{d}(Tu)\|_{t,B}\left(\int_B w^{t\alpha/(t-s)}\,\mathrm{d}x\right)^{(t-s)/st}\\ &\leqslant \|\mathrm{d}(Tu)\|_{t,B}\left(\int_B w\,\mathrm{d}x\right)^{\alpha/s}\\ &\leqslant (\|u\|_{t,B}+C_2\,\mathrm{diam}(B)\|\mathrm{d}u\|_{t,B})\left(\int_B w\,\mathrm{d}x\right)^{\alpha/s}\\ &\leqslant (\|u\|_{t,B}+C_3\|u\|_{t,\sigma B})\left(\int_B w\,\mathrm{d}x\right)^{\alpha/s}\\ &\leqslant C_4\|u\|_{t,\sigma B}\left(\int_B w\,\mathrm{d}x\right)^{\alpha/s}\\ &\leqslant C_5|B|^{(m-t)/mt}\|u\|_{m,\sigma B}\left(\int_B w\,\mathrm{d}x\right)^{\alpha/s}.\end{aligned}\tag{2.24}$$

Using Lemma 2.3, we obtain

$$\begin{aligned}\|u\|_{m,\sigma B} &= \left(\int_{\rho B}(|u|w^{\alpha/s}w^{-\alpha/s})^m \,\mathrm{d}x\right)^{1/m}\\ &\leqslant \left(\int_{\rho B}|u|^s w^{\alpha}\,\mathrm{d}x\right)^{1/s}\left(\int_{\rho B}\left(\frac{1}{w}\right)^{1/(r-1)}\mathrm{d}x\right)^{\alpha(r-1)/s} \qquad (2.25)\end{aligned}$$

for all balls B with $\sigma B \subset \Omega$. Substituting (2.25) into (2.24), then using (2.13) (replacing ρ by σ in (2.13)), we obtain

$$\left(\int_B |\mathrm{d}(Tu)|^s w^{\alpha}\,\mathrm{d}x\right)^{1/s} \leqslant C_6|B|^{(1-\alpha)/s}\left(\int_{\sigma B}|u|^s w^{\alpha}\,\mathrm{d}x\right)^{1/s}$$

which ends the proof of Theorem 2.21 for the case $0 < \alpha < 1$. For the case $\alpha = 1$, the proof is similar to that of Theorem 2.6. □

Note that the parameter α in both of Theorem 2.6 and Theorem 2.21 is any real number with $0 < \alpha \leqslant 1$. Therefore, we can have different versions of the weighted imbedding inequality by choosing α to be different values. For example, set $t = 1-\alpha$ in Theorem 2.6 and write $\mathrm{d}\mu = w(x)\,\mathrm{d}x$. Then, inequality (2.7) becomes

$$\left(\int_B |Tu|^s w^{-t}\,\mathrm{d}\mu\right)^{1/s} \leqslant C\,\mathrm{diam}(B)\left(\int_{\rho B}|u|^s w^{-t}\,\mathrm{d}\mu\right)^{1/s}. \qquad (2.26)$$

If we choose $\alpha = 1/r$ in Theorem 2.6, then (2.7) reduces to

$$\left(\int_B |Tu|^s w^{1/r}\,\mathrm{d}x\right)^{1/s} \leqslant C\,\mathrm{diam}(B)\left(\int_{\rho B}|u|^s w^{1/r}\,\mathrm{d}x\right)^{1/s}. \qquad (2.27)$$

If we choose $\alpha = 1/s$ in Theorem 2.6, then $0 < \alpha < 1$ since $1 < s < \infty$. Thus, (2.7) reduces to the following symmetric version:

$$\left(\int_B |Tu|^s w^{1/s}\,\mathrm{d}x\right)^{1/s} \leqslant C\,\mathrm{diam}(B)\left(\int_{\rho B}|u|^s w^{1/s}\,\mathrm{d}x\right)^{1/s}. \qquad (2.28)$$

Finally, if we choose $\alpha = 1$ in Theorem 2.6, we have the following weighted imbedding inequality.

$$\|Tu\|_{s,B,w} \leqslant C\,\mathrm{diam}(B)\|u\|_{s,\rho B,w}. \qquad (2.29)$$

REMARK. Choosing α to be some special values in Theorem 2.21, we shall have some similar results. For example, selecting $\alpha = 1$ in Theorem 2.21, we have

$$\|\mathrm{d}(Tu)\|_{s,B,w} \leqslant C_1|B|^{(1-\alpha)/s}\|u\|_{s,\sigma B,w} \leqslant C_2\|u\|_{s,\sigma B,w}. \qquad (2.30)$$

3. Global Weighted Imbedding Inequalities

We need the following properties of the Whitney covers appearing in [8] to prove the global result.

LEMMA 3.1. *Each Ω has a modified Whitney cover of cubes $\nu = \{Q_i\}$ such that*

$$\bigcup_i Q_i = \Omega,$$

$$\sum_{Q\in\nu} \chi_{(\sqrt{\frac{5}{4}})Q} \leqslant N_{\chi\Omega}$$

for all $x \in \mathbf{R}^n$ and some $N > 1$ and if $Q_i \cap Q_j \neq \phi$, then there exists a cube R (this cube does not need be a member of ν) in $Q_i \cap Q_j$ such that $Q_i \cup Q_j \subset NR$. Moreover if Ω is δ-John, then there is a distinguished cube $Q_0 \in \nu$ which can be connected with every cube $Q \in \nu$ by a chain of cubes $Q_0, Q_1, \ldots, Q_k = Q$ from ν and such that $Q \subset \rho Q_i$, $i = 0, 1, 2, \ldots, k$, for some $\rho = \rho(n, \delta)$.

We prove the following global $A_r(\Omega)$-weighted imbedding inequality in a bounded domain Ω for A-harmonic tensors.

THEOREM 3.2. *Let $u \in L^s(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be an A-harmonic tensor in a bounded domain $\Omega \subset \mathbf{R}^n$ and $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be a homotopy operator defined by*

$$Tw = \int_\Omega \varphi(y) K_y w \, \mathrm{d}y.$$

Assume that $w \in A_r(\Omega)$ for some $r > 1$. Then, there exists a constant C, independent of u, such that

$$\left(\int_\Omega |Tu|^s w^\alpha \, \mathrm{d}x\right)^{1/s} \leqslant C\left(\int_\Omega |u|^s w^\alpha \, \mathrm{d}x\right)^{1/s}, \tag{3.3}$$

$$\left(\int_w |\mathrm{d}Tu|^s w^\alpha \, \mathrm{d}x\right)^{1/s} \leqslant C\left(\int_\Omega |u|^s w^\alpha \, \mathrm{d}x\right)^{1/s} \tag{3.4}$$

for any real number α with $0 < \alpha \leqslant 1$.

Proof. Using (2.7) and Lemma 3.1, we have

$$\begin{aligned}\left(\int_\Omega |Tu|^s w^\alpha \, \mathrm{d}x\right)^{1/s} &\leqslant \sum_{Q\in\nu}\left(C_1 \operatorname{diam}(Q)\left(\int_{\rho Q} |u|^s w^\alpha \, \mathrm{d}x\right)^{1/s}\right) \\ &\leqslant C_1 \operatorname{diam}(\Omega) \sum_{Q\in\nu}\left(\int_{\rho Q} |u|^s w^\alpha \, \mathrm{d}x\right)^{1/s}\end{aligned}$$

$$\leqslant C_1 \operatorname{diam}(\Omega) \sum_{Q \in \nu} \left(\int_\Omega |u|^s w^\alpha \, dx \right)^{1/s}$$

$$\leqslant C_3 \left(\int_\Omega |u|^s w^\alpha \, dx \right)^{1/s}$$

which indicates that (3.3) holds. Using (2.22) and Lemma 3.1, we can prove (3.4) similarly. The proof of Theorem 3.2 has been completed. □

REMARK. Choosing a to be some special values in (3.3) and (3.4), we shall have some global results similar to the local case.

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